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# Four sets of $_3F_2(1)$ functions, Hahn polynomials and recurrence relations for the 3-$j$ coefficients

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**Abstract.** The Wigner, Racah and Majumdar sets of $_3F_2(1)$ are derived from the symmetric van der Waerden set of $_3F_2(1)$ for the 3-$j$ coefficient, using a transformation due to Erdelyi and Weber, which is also used to relate the 3-$j$ coefficient to a discrete orthogonal Hahn polynomial. The four recurrence relations (one old and three new) obtained by Karlin and McGregor for the Hahn polynomial are used to derive recurrence relations for the 3-$j$ coefficient, two of which have been shown to be useful in the exact recursive evaluation of the 3-$j$ coefficients by Schulten and Gordon.

## 1. Introduction

In the literature (see Smorodinskii and Shelepin 1972), the 3-$j$ coefficient has been related to a $_3F_2(1)$ by Wigner (1940), Racah (1942), van der Waerden (1932) and Majumdar (1955). Starting from the van der Waerden form and resorting to the work of Whipple (1925) on the symmetries of the $_3F_2(1)$ functions, Raynal (1978) has shown that ten different forms of $_3F_2(1)$ can be obtained. One of us has shown (Srinivasa Rao 1978a) that a set of six $_3F_2(1)$ of the van der Waerden type is necessary and sufficient to account for the 72 symmetries of the 3-$j$ coefficient. Here we show that, starting with the highly symmetric van der Waerden set of six $_3F_2(1)$, three equivalent sets of $_3F_2(1)$ corresponding to the Wigner, Racah and Majumdar forms can be derived by simply using a Weber-Erdelyi (1952) transformation in three different ways. We discuss the symmetries of the 3-$j$ coefficient in terms of these sets of $_3F_2(1)$.

Recently, there has been considerable interest in unravelling the deep connection between the basic quantities of the quantum theory of angular momentum, viz the Clebsch-Gordan (or 3-$j$) coefficients and the Racah (or 6-$j$) coefficients and orthogonal polynomials of a discrete variable (i.e. polynomials which are orthogonal on a discrete set of points). Smorodinskii and Suslov (1982), while determining the eigenvalues and eigenvectors of a Hermitian operator, were led to a relation between 3-$j$ coefficients and discrete orthogonal Hahn polynomials 'which are practically unknown to physicists'. Wilson (1980) and Askey and Wilson (1979) related the 6-$j$ coefficient to the orthogonal polynomial called the Racah polynomial, which contains as limiting cases the classical polynomials of Jacobi, Laguerre and Hermite and their discrete analogues which go under the names of Hahn, Meixner, Krawtchouk and Charlier polynomials. Askey and Wilson (1979) discuss the classical type of orthogonal polynomials that can be given as hypergeometric polynomials and they also provide a chart showing their interrelationship.

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Schulten and Gordon (1975) realised the need for the evaluation of whole strings of 3-$j$ coefficients (rather than the evaluation of a single coefficient) of two different types and provided a numerical algorithm based on the recursion equations relating coefficients in the strings. They derived the recursion relations algebraically from certain sum rules satisfied by the 3-$j$ coefficients and they state that 'while this derivation is the shortest available, it is somewhat remote from the definitions of the coefficients'.

Here, we relate the 3-$j$ coefficient to the van der Waerden form of the ${}_3F_2(1)$ functions, transform it to the Majumdar form of ${}_3F_2(1)$ using a Weber-Erdelyi transformation and then relate the 3-$j$ coefficient to the discrete orthogonal Hahn polynomial. Karlin and McGregor (1961), in their classic paper entitled *The Hahn Polynomials, Formulas and an Application*, have provided a complete list of recurrence relations satisfied by the orthogonal Hahn and dual Hahn polynomial. Of the four recurrence relations proved by Karlin and McGregor three are noted by them to be new. Having established that the discrete orthogonal Hahn polynomial is related to the 3-$j$ coefficient, we study the consequences of the aforesaid recurrence relations on the 3-$j$ coefficients themselves. We are thus led to a simple and straightforward derivation of four recurrence relations, which include the two derived from sum rules by Schulten and Gordon (1975). Since one of these recurrence relations is a linear combination of two others, it follows that three of the four recurrence relations are independent. Of these three, two recurrence relations satisfied by the 3-$j$ coefficient are new.

## 2. Four sets of ${}_3F_2(1)$ functions

Several methods used for calculating the 3-$j$ coefficients have been summarised by Biedenharn and Louck (1981). The 3-$j$ coefficient is defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= \delta(m_1+m_2+m_3, 0)(-1)^{j_1-j_2-m_3}\Delta(j_1j_2j_3)\prod_{i=1}^{3}[(j_i+m_i)!(j_i-m_i)!]^{1/2}$$

$$\times\sum_t(-1)^t\left(t!\prod_{k=1}^{2}(t-\alpha_k)!\prod_{l=1}^{3}(\beta_l-t)!\right)^{-1} \tag{1}$$

where

$$\max(\alpha_1,\alpha_2)\leqslant t\leqslant\min(\beta_1,\beta_2,\beta_3)$$

$$\beta_1=j_1-m_1 \qquad \beta_2=j_2+m_2 \qquad \beta_3=j_1+j_2-j_3$$

$$\alpha_1=j_1-m_1-(j_3+m_3) \qquad \alpha_2=j_2+m_2-(j_3-m_3)$$

and

$$\Delta(xyz)=[(-x+y+z)!(x-y+z)!(x+y-z)!/(x+y+z+1)!]^{1/2}.$$

This form is the most symmetric form attributed to van der Waerden (1932) and it has also been arrived at by Racah (1942).

One of us (Srinivasa Rao 1978a) has shown that there exist a set of six series representations, and correspondingly a set of six ${}_3F_2(1)$ functions, necessary and sufficient to account for the well known 72 symmetries of the 3-$j$ coefficient. This set of six ${}_3F_2(1)$ functions will be referred to as the van der Waerden set and explicitly it is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1+m_2+m_3, 0)(-1)^{\alpha(pqr)} \prod_{i,k=1}^{3} \{R_{ik}!/(J+1)!\}^{1/2}$$
$$\times [\Gamma(1-A, 1-B, 1-C, D, E)]^{-1} {}_3F_2(A, B, C; D, E; 1) \tag{2}$$

where

$$A = -R_{2p} \qquad B = -R_{3q} \qquad C = -R_{1r}$$
$$D = R_{3r} - R_{2p} + 1 \qquad E = R_{2r} - R_{3q} + 1$$
$$\Gamma(x, y, \ldots) = \Gamma(x)\Gamma(y)\ldots \qquad J = j_1 + j_2 + j_3$$

and

$$\alpha(pqr) = \begin{cases} R_{3p} - R_{2q} & \text{for even permutations,} \\ R_{3p} - R_{2q} + J & \text{for odd permutations} \end{cases}$$

for all permutations of $(pqr) = (123)$. The $R_{ik}$ are the elements of the $3\times3$ square array given by Regge (1958):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{Vmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{Vmatrix} \tag{3}$$

whose row and column sums add up to $J = j_1+j_2+j_3$ and which exhibits, due to its invariance under column permutations, row permutations and transposition, the 72 symmetries of the 3-$j$ coefficient.

We now make use of a transformation formula for a terminating ${}_3F_2(1)$. This formula is one of a group (cf Bailey 1935) and its proof, given by Weber and Erdelyi (1952), runs along the following lines. The formula

$${}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = \frac{\Gamma(\delta)}{\Gamma(\beta, \delta-\beta)} \int_0^1 {}_2F_1(-n, \alpha; \gamma; t) t^{\beta-1}(1-t)^{\delta-\beta-1}\, \mathrm{d}t \tag{4}$$

can be verified by expanding both sides in power series. We use the well known identity:

$${}_2F_1(-n, \alpha; \gamma; t) = \frac{\Gamma(\gamma, \gamma+n-\alpha)}{\Gamma(\gamma+n, \gamma-\alpha)} {}_2F_1(-n, \alpha; \alpha-n-\gamma+1; 1-t) \tag{5}$$

and substitute it in (4). Replacing the variable $t$ by $1-t$ and using (4) again to replace the integral with a ${}_3F_2(1)$ we get the transformation formula:

$${}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = \frac{\Gamma(\gamma, \gamma+n-\alpha)}{\Gamma(\gamma+n, \gamma-\alpha)} {}_3F_2(-n, \alpha, \delta-\beta; 1+\alpha-\gamma-n, \delta; 1) \tag{6}$$

where $n$ is an integer which determines the number of terms in the ${}_3F_2(1)$. We refer to (6) as the Erdelyi-Weber (EW) transformation formula.

Identifying the numerator and denominator parameters of the van der Waerden set of ${}_3F_2(1)$ functions given in (2) as

$$\alpha = A \qquad \beta = B \qquad n = -C \qquad \gamma = D \qquad \delta = E \tag{7}$$

and applying the EW transformation (6), we will get for the 3-$j$ coefficient (in the notation adopted for (2)):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(m_1+m_2+m_3, 0)(-1)^{\alpha(pqr)} \prod_{i,k=1}^{3} \{R_{ik}!/(J+1)!\}^{1/2}$$
$$\times\Gamma(1-D')\{\Gamma(1-A', 1+B'-E', 1-C', E', 1+A'-D', 1+C'-D')\}^{-1}$$
$$\times {}_3F_2(A', B', C'; D', E'; 1) \tag{8}$$

where

$$A' = -R_{2p} \qquad B' = 1+R_{2r} \qquad C' = -R_{1r}$$
$$D' = -R_{1r} - R_{3r} \qquad E' = R_{2r} - R_{3q} + 1. \tag{9}$$

This set of ${}_3F_2(1)$ functions will be called the Wigner set of ${}_3F_2(1)$, since in (9), setting $(pqr) = (132)$ results in the Wigner form of the 3-$j$ coefficient given by equation (28) in Raynal (1978).

Alternatively, if we identify the parameters in (2) as

$$\alpha = A \qquad \beta = C \qquad n = -B \qquad \gamma = D \qquad \delta = E \tag{10}$$

and using the EW transformation (6), we will get for the 3-$j$ coefficient the form (8), but the numerator and denominator parameters of the $3F_2(1)$ will now be

$$A' = -R_{2p} \qquad B' = 1+R_{3p} \qquad C' = -R_{3p}$$
$$D' = -R_{3p} - R_{3r} \qquad E' = 1+R_{2r} - R_{3q}. \tag{11}$$

This set will be called the Racah set of ${}_3F_2(1)$ functions, since in (11), identifying $(pqr) = (132)$, the Racah form of the 3-$j$ coefficient, viz equation (29) in Raynal (1978), can be obtained. Biedenharn and Louck (1981) in their treatise point out that Racah's form may be obtained from Wigner's form by using the two transformations that arise due to interchanging the second and third rows of (3) followed by the interchange of the first and second rows of (3).

Finally, a third identification for the parameters in (2) as

$$\alpha = C \qquad \beta = A \qquad n = -B \qquad \gamma = D \qquad \delta = E \tag{12}$$

and the use of (6) will yield for the 3-$j$ coefficient the form of (8) but with the numerator and denominator parameters being

$$A' = -R_{1r} \qquad B' = 1+R_{1q} \qquad C' = -R_{3q}$$
$$D' = -R_{2q} - R_{3q} \qquad E' = 1+R_{2r} - R_{3q}. \tag{13}$$

This set of ${}_3F_2(1)$ functions will be called the Majumdar set, since for $(pqr) = (321)$ the Majumdar form of the 3-$j$ coefficient given by equation (30) in Raynal (1978) is obtained.

Thus it is found that, starting with the highly symmetric van der Waerden set of ${}_3F_2(1)$, three sets of ${}_3F_2(1)$ corresponding to Wigner, Racah and Majumdar forms can be obtained by simply using the Erdelyi-Weber transformation in three different ways. Conversely, the same EW transformation can be used to get the van der Waerden set from the Wigner, Racah or Majumdar sets, by virtue of the fact that the matrix relating the numerator and denominator parameters in (6) acts like a projection operator.

Corresponding to the three identifications made above—viz (7), (10) and (12)—we can make three more identifications with

$$\gamma = E \qquad \delta = D \tag{14}$$

when we again get the three sets but in a different order, viz Majumdar, Racah and Wigner forms of ${}_3F_2(1)$ given by (13), (11) and (9), respectively, on which are superposed (i) the interchange of the $p$, $q$ indices and (ii) the $m_i \to -m_i$ substitution.

While we have obtained here four sets of ${}_3F_2(1)$ for the 3-$j$ coefficient, one member of the Wigner, Racah, van der Waerden and Majumdar ${}_3F_2(1)$ forms has been referred to by Smorodinskii and Shelepin (1972). Also, starting with a given ${}_3F_2(1)$ belonging to the van der Waerden set and resorting to the work of Whipple (1925) on the symmetries of the ${}_3F_2(1)$ functions, Raynal (1978) has shown that the ${}_3F_2(1)$ forms due to Wigner, Racah and Majumdar can be obtained.

The following is to be noted. In the case of the van der Waerden set of six ${}_3F_2(1)$, all the 3! numerator parameter permutations and the 2! denominator parameter permutations are allowed, as is manifestly evident from (2). For each member of the set, these permutations account for 12 symmetries, and hence for the whole set all 72 symmetries of the 3-$j$ coefficient will be accounted for. But, in the case of the Wigner, Racah and Majumdar sets of six ${}_3F_2(1)$, each member of the ${}_3F_2(1)$ set accounts for only two symmetries (and not all 12 as one would expect). This is due to the nature of the numerator and denominator parameters. In the case of the van der Waerden set all three numerator parameters are negative integer parameters and the two denominator parameters are positive integers. But in the case of the Wigner, Racah and Majumdar sets, two of the three numerator parameters ($A'$ and $C'$) are negative integers while the third ($B'$) is a positive integer and of the two denominator parameters one ($D'$) is always a negative integer and the other ($E'$) is always a positive integer. Amongst the numerator/denominator parameters, permutation of negative (or positive) integer parameters will yield meaningful and known symmetries of the 3-$j$ coefficient. But permutation of a negative parameter with a positive parameter (in the numerator/denominator) will yield symmetries for the 3-$j$ coefficient which violate the triangular inequalities as in the case of the 6-$j$ coefficient obtained by Minton (1970). To illustrate, in the Wigner ${}_3F_2(1)$ set, given by (8) and (9) for $(pqr) = (132)$, interchanging $B'$ and $C'$ will result in the 3-$j$ coefficient being related to

$$\begin{pmatrix} (j_1-j_3+m_2-1)/2 & j_2 & (-j_1+j_3+m_2-1)/2 \\ m_1+(-j_1-j_3+m_2-1)/2 & j_1+j_3+1 & m_3+(-j_1-j_3+m_2-1)/2 \end{pmatrix} \tag{15}$$

which, though a Regge-like symmetry in appearance, violates the triangular inequality (for the 3-$j$ coefficient on the right-hand side of (15)). Thus, the only allowed symmetries in the Wigner, Racah and Majumdar sets of ${}_3F_2(1)$ are those due to the interchange of $A'$ and $C'$, as is manifestly evident from the form of (8). The asymmetric nature of these forms has been realised by Racah (1942) himself as reflected in his statement that his formula 'is similar to Wigner's formula and is, also, unsymmetrical and unpractical for the use'. Racah (1942) transformed his formula into the highly symmetrical van der Waerden form. One of us (Srinivasa Rao 1978b, 1981) has shown how the van der Waerden set of ${}_3F_2(1)$ is most useful in numerical computation of the 3-$j$ coefficient.

## 3. The Hahn polynomial: definition and properties

The Hahn polynomials defined by Karlin and McGregor (1961) are

$$\begin{aligned} Q_n(x) &\equiv Q_n(x;\alpha,\beta,N) \\ &= {}_3F_2(-n,-x,n+\alpha+\beta+1;\alpha+1,-N+1;1) \end{aligned} \tag{16}$$

for real $\alpha > -1$, $\beta > -1$ and positive integral $N$. The results of Karlin and McGregor, which are made use of here, are obtained with this restriction of $\alpha$, $\beta$ to real values $> -1$.

This discrete polynomial has been shown (Karlin and McGregor 1961) to satisfy the following orthogonality relations:

$$\sum_{x=0}^{N-1} Q_n(x) Q_m(x) \rho(x) = \frac{1}{\pi_n} \delta_{m,n} \tag{17}$$

and

$$\sum_{n=0}^{N-1} Q_n(x) Q_n(y) \pi_n = \frac{1}{\rho(x)} \delta_{x,y} \tag{18}$$

where $\delta_{x,y}$ is the Kronecker delta function and the weight functions are

$$\rho(x) = \rho(x;\, \alpha, \beta, N) = \binom{\alpha+x}{x} \binom{\beta+N-1-x}{N-1-x} \binom{N+\alpha+\beta}{N-1}^{-1} \tag{19}$$

and

$$\begin{aligned} \pi_n &= \pi_n(\alpha, \beta, N) \\ &= \binom{N-1}{n} \binom{N+\alpha+\beta+n}{n}^{-1} \frac{(2n+\alpha+\beta+1)}{(\alpha+\beta+1)} \\ &\quad \times \frac{\Gamma(\beta+1,\, n+\alpha+1,\, n+\alpha+\beta+1)}{\Gamma(\alpha+1,\, \alpha+\beta+1,\, n+\beta+1,\, n+1)} \end{aligned} \tag{20}$$

with $\binom{n}{r}$ representing the usual binomial coefficients. Karlin and McGregor call (18) their new dual orthogonality relation.

To relate the 3-$j$ coefficient given by $(pqr) = (123)$ in (2) to the Hahn polynomial, we make use of the transformation (6) for the ${}_3F_2(1)$ given by Erdelyi and Weber. Identifying

$$\alpha = C \qquad \beta = A \qquad n = -B \qquad \gamma = D \qquad \delta = E \tag{21}$$

after simplification, we get

$$\begin{aligned} Q_n(x) &= (-1)^{2j_2+m_1+n+x} \frac{(j_3-j_2-m_1)!}{(2j_2)!} \\ &\quad \times \left( \frac{(j_2-n)!\, n!\, (2j_3+n+1)!}{(2(j_3-j_2)+n)!\, (j_3-j_2+m_1+n)!} \, \frac{x!\, (2j_2-x)!\, (j_3-j_2-m_1+n)!}{(j_3-j_2+m_1+x)!\, (j_3+j_2-m_1-x)!} \right)^{1/2} \\ &\quad \times \begin{pmatrix} j_3-j_2+n & j_2 & j_3 \\ m_1 & x-j_2 & j_2-m_1-x \end{pmatrix} \end{aligned} \tag{22}$$

where we set

$$n = j_1+j_2-j_3 \qquad x = j_2+m_2 \qquad N = 2j_2+1$$

$$\alpha = j_3-j_2+m_1 \qquad \beta = -j_2+j_3-m_1 .$$

Though $\alpha = (j_3 - m_3) - (j_2 + m_2)$ and $\beta = (j_3 + m_3) - (j_2 - m_2)$, being differences between integer quantities, appear to be capable of taking positive or negative values, due to the 72 symmetries of the 3-$j$ coefficient, it is always possible to choose a symmetry of the given 3-$j$ coefficient for which both $\alpha$ and $\beta$ are $\geqslant 0$. This restriction to non-negative real values of $\alpha$ and $\beta$ is required since we use the orthogonality properties for the Hahn and dual Hahn polynomials of Karlin and McGregor (1961).

## 4. Recurrence relations

The first of the recurrence relations satisfied by the Hahn polynomial due to Weber and Erdelyi (1952) is

$$[b_n + d_n - x]Q_n(x) = b_n Q_{n+1}(x) + d_n Q_{n-1}(x) \tag{23}$$

where

$$b_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n-1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \tag{24}$$

$$d_n = \frac{n(n+\beta)(n+\alpha+\beta+N)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \tag{25}$$

which is valid for complex values of $x$ if $n = 0, 1, 2, \ldots, N-2$ but is valid only for $x = 0, 1, 2, \ldots, N-1$ when $n = N-1$. Using (22), (24), and (25) in (23) after simplifying and rearranging we get the following recurrence relation for the 3-$j$ coefficient:

$$B(j_1, j_2, j_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + (j_1+1)A(j_1, j_2, j_3)\begin{pmatrix} j_1-1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$
$$+ j_1 A(j_1+1, j_2, j_3)\begin{pmatrix} j_1+1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \tag{26}$$

where

$$A(j_p, j_q, j_r) = [j_p^2 - (j_q - j_r)^2]^{1/2}[-j_p^2 + (j_q + j_r + 1)^2]^{1/2}[j_p^2 - m_p^2]^{1/2} \tag{27}$$

$$B(j_p, j_q, j_r) = (2j_p+1)\{j_p(j_p+1)(m_r - m_q) - [j_q(j_q+1) - j_r(j_r+1)]m_p\} \tag{28}$$

with $p \neq q \neq r$ being 1, 2 or 3. These expressions, with minor notational modifications, correspond to (6*a*), (6*b*) and (6*c*) of Schulten and Gordon (1975), respectively.

The orthogonality relation (18) for the discrete Hahn polynomial can be shown to imply the following normalisation condition for the 3-$j$ coefficient:

$$\sum_{j_1} (2j_1+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1. \tag{29}$$

Schulten and Gordon (1975) have provided a numerical algorithm for the computation of the 3-$j$ coefficient based on recursion equations relating coefficients in two different types of strings. They derived the recursion relations algebraically from certain sum rules satisfied by these coefficients. The orthogonality relation (29), along with the recurrence relation (26), has been shown by them to be adequate to determine (except for an overall phase factor) the values of the string of 3-$j$ coefficients $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ for all allowed values of $j_1$.

The second difference equation derived by Karlin and McGregor for the Hahn polynomial is

$$[B(x)+D(x)-\lambda_n]Q_n(x)=B(x)Q_n(x+1)+D(x)Q_n(x-1) \tag{30}$$

where

$$B(x)=(N-1-x)(\alpha+1+x)$$

$$D(x)=x(N+\beta-x)$$

$$\lambda_n=n(n+\alpha+\beta+1)$$

and (30) is valid for $n=0, 1, \ldots, N-1$, for all complex values of $x$. This recurrence relation implies for the 3-$j$ coefficient:

$$C(m_2+1, m_3-1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix}+D(m_2, m_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$+C(m_2, m_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}=0 \tag{31}$$

where

$$C(m_p, m_q)=[(j_p-m_p+1)(j_p+m_p)(j_q-m_q)(j_q+m_q+1)]^{1/2} \tag{32}$$

and

$$D(m_p, m_q)=-j_r(j_r+1)+j_p(j_p+1)+j_q(j_q+1)+2m_pm_q \tag{33}$$

with $p\neq q\neq r$ being 1, 2 or 3. These expressions correspond to the appropriately modified forms of (9$a$), (9$b$) and (9$c$) of Schulten and Gordon (1975). The orthogonality relation (17) can be shown to imply the normalisation condition:

$$\sum_{m_2}(2j_1+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2=1 \tag{34}$$

which, along with the recurrence relation (31), has been shown by Schulten and Gordon to determine (except for an overall phase factor) the values of the string of 3-$j$ coefficients $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix}$ for all allowed values of $m_2$.

Thus, the recurrence relation in $j_1$ and the recurrence relation in $m_2$ and $m_3$ are found to be direct consequences of the corresponding recurrence relations satisfied by the discrete orthogonal Hahn polynomials. The derivations of (26) and (31) given here are a direct consequence of the definition of the 3-$j$ coefficient in terms of $Q_n(x)$ given in (22), as opposed to the algebraic method resorted to by Schulten and Gordon of deriving them from certain other sum rules.

Karlin and McGregor (1961) have given two new first-order difference-recurrence relations satisfied by the Hahn polynomial. These are

$$\{(n+\alpha+\beta+1)[(n+\beta+1)(x-n)-(n+\alpha+1)(N-1-x)]$$

$$+(2n+\alpha+\beta+2)(\alpha+1+x)(N-1-x)\}Q_n(x)$$

$$-(2n+\alpha+\beta+2)(\alpha+1+x)(N-1-x)Q_n(x+1)$$

$$+(n+\alpha+\beta+1)(n+\alpha+1)(N-1-n)Q_{n+1}(x)=0 \tag{35}$$

and

$$\{n[(n+\beta)(N-1-x)-(n+\alpha)(n+\alpha+\beta+x+1)] \\ +(2n+\alpha+\beta)(\alpha+1+x)(N-1-x)\}Q_n(x) \\ -(2n+\alpha+\beta)(\alpha+1+x)(N-1-x)Q_n(x+1) \\ -n(n+\beta)(n+\alpha+\beta+N)Q_{n-1}(x)=0. \tag{36}$$

While (23) is a three-term recurrence relation in $n$ for $Q_n(x)$ and (30) is a three-term recurrence relation in $x$ for $Q_n(x)$, it is to be noted that (35) and (36) are recurrence relations mixed in $n$ and $x$. However, since a term involving $Q_n(x+1)$ is common in both (35) and (36), one can try to algebraically eliminate it. This results in (23)—a three-term recurrence relation in $n$. Therefore, we consider (35) and (36) along with (30) to be the fundamental recurrence relations satisfied by $Q_n(x)$.

A straightforward use of (22) in (35) and (36), after simplification and rearrangement, leads to the following recurrence relations for the 3-$j$ coefficient:

$$F(j_1,j_2,j_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}+2(j_1+1)C(m_3,m_2)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} \\ -A(j_1+1,j_2,j_3)\begin{pmatrix} j_1+1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}=0 \tag{37}$$

and

$$E(j_1,j_2,j_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}+2j_1C(m_3,m_2)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} \\ +A(j_1,j_2,j_3)\begin{pmatrix} j_1-1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}=0 \tag{38}$$

where

$$F(j_p,j_q,j_r)=(j_p-j_q+j_r+1)[(j_p+1)(j_p+j_q-j_r-2m_q)+m_p(-j_p+j_q+j_r)] \tag{39}$$

and

$$E(j_p,j_q,j_r)=(-j_p-j_q+j_r)[j_p(j_p-j_q+j_r+2m_q+1)+m_p(j_p+j_q+j_r+1)]. \tag{40}$$

Multiplying (37) by $j_1$ and (38) by $(j_1+1)$ and subtracting, we would get the three-term recurrence relation in $j_1$ for the 3-$j$ coefficient, with the constant factors obeying the condition

$$(j_1+1)E(j_1,j_2,j_3)-j_1F(j_1,j_2,j_3)=B(j_1,j_2,j_3). \tag{41}$$

To conclude, we have shown in § 2 that, just as a set of six ${}_3F_2(1)$ exist for the van der Waerden case, there exist sets of six ${}_3F_2(1)$ for the Wigner, Racah and Majumdar forms. We have also established an interrelationship between these sets of ${}_3F_2(1)$ with the help of the Erdelyi–Weber transformation (6).

In § 4, we have shown that, as a direct consequence of identifying the 3-$j$ coefficient with a discrete orthogonal Hahn polynomial, we can derive three fundamental recurrence relations for the 3-$j$ coefficient and two of these are new. Smorodinskii and Suslov (1982) made a different identification (see the appendix) but that has also been found to lead to the same three recurrence relations—(31), (37) and (38)—for the 3-$j$ coefficient.

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## Appendix

The transformation (6) given by Erdelyi and Weber can, in fact, be derived from tables $\mathrm{II}_A$ and $\mathrm{II}_B$ in Bailey (1935) which summarise and group the equivalent numerator and denominator parameters of the ${}_3F_2(1)$ functions obtained by Thomae in the notation introduced by Whipple (1925). Explicitly, in this notation, (6) corresponds to

$$F_p(0; 4, 5) = (-1)^m \frac{\Gamma(\alpha_{124}, \alpha_{024}, \alpha_{014})}{\Gamma(\alpha_{123}, \alpha_{124}, \alpha_{125})} F_n(4; 0, 1). \tag{A1}$$

Using (6) again, with the roles of $\gamma$ and $\delta$ interchanged, to transform the right-hand side of (6), Erdelyi and Weber obtained the transformation:

$$\begin{aligned}&{}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1)\\ &\quad= \frac{\Gamma(\gamma, \delta, \delta+n-\alpha, \gamma+n-\alpha)}{\Gamma(\gamma+n, \delta+n, \delta-\alpha, \gamma-\alpha)}\, {}_3F_2\left(\begin{matrix}-n, \alpha, 1+\alpha+\beta-\gamma-\delta-n; 1\\ 1+\alpha-\delta-n, 1+\alpha-\gamma-n\end{matrix}\right).\end{aligned} \tag{A2}$$

If we identify

$$\alpha = A \qquad \beta = B \qquad n = -C \qquad \gamma = D \qquad \delta = E \tag{A3}$$

where $A$, $B$, $C$; $D$, $E$ are the numerator and denominator parameters of the van der Waerden form of ${}_3F_2(1)$ given by $(pqr) = (123)$ in (2), and follow the procedure outlined in the text, then we would get

$$\begin{aligned}Q_n(x) &= \frac{(-1)^{j_1-j_2-m_3}}{(2j_1)!(j_1+j_2+m_3)!} \left\{\frac{n!(2j_1-n)!(2(j_1+j_2)-n+1)!}{(2j_2-n)!(j_1+j_2-m_3-n)!}\right\}^{1/2}\\ &\quad\times [x!(2j_1-x)!(j_1+j_2+m_3-x)!(-j_1+j_2-m_3+x)!(j_1+j_2+m_3-n)!]^{1/2}\\ &\quad\times \begin{pmatrix} j_1 & j_2 & j_1+j_2-n\\ j_1-x & -j_1-m_3+x & m_3\end{pmatrix}\end{aligned} \tag{A4}$$

for the discrete Hahn polynomial (16), with:

$$\begin{aligned}&n = j_1+j_2-j_3 \qquad x = j_1-m_1 \qquad N = 2j_1+1\\ &\alpha = -j_1-j_2-m_3-1 \qquad \beta = -j_1-j_2+m_3-1.\end{aligned} \tag{A5}$$

This form (A4) happens to be an equivalent way of relating the Hahn polynomial to the 3-$j$ coefficient and is similar to that given by Smorodinskii and Suslov (1982), who also made use of (A2).

In passing, we wish to mention that the identification (21) made in the text transforms the van der Waerden ${}_3F_2(1)$ form for the 3-$j$ coefficient, via the Erdelyi–Weber transformation (6), to the Majumdar form (13). If, instead of (21), we make the identifications (7) and (10), then we would have obtained, after the Erdelyi–Weber transformation (6), the ${}_3F_2(1)$ forms (9) or (11), which are the Wigner and Racah ${}_3F_2(1)$ forms for the 3-$j$ coefficient as detailed in § 2. However, these two identifications do not lead to the desired ranges for the indices $x$ and $n$ to satisfy the known sum rules for the 3-$j$ coefficient given in (29) and (34).

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